Quick Computation of Spatial Autoregressive Estimators

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ABSTRACT:

Spatial estimators usually require the manipulation of n^2 relations among n observations and use operations such as determinants, eigenvalues, and inverses whose operation counts grow at a rate proportional to n^3. This paper provides ways to quickly compute estimates when the dependent variable follows a spatial autoregressive process, which by appropriate specification of the independent variables can subsume the case when the errors follow a spatial autoregressive process. Since only nearby observations tend to affect a given observation, most observations have no effect and hence the spatial weight matrix becomes sparse. By exploiting sparsity and rearranging computations, one can compute estimates at low cost. As a demonstration of the efficacy of these techniques, the paper provides a Monte Carlo study whereby 3107 observation regressions require only 0.1 seconds each when using Matlab on a 200Mhz Pentium Pro personal computer. In addition, the paper illustrates these techniques by examining voting behavior across US counties in the 1980 presidential election.

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INTRODUCTION

The examination of empirical data over space, with explicit recognition of the influence each observation has upon the others, has made large gains since the seminal contributions of Whittle (1954). For example, Ord (1975) proposed an algorithm involving eigenvalues which made spatial estimation practical for small to moderate sized data sets. In addition, Griffith (1988), Anselin (1988), Haining (1990), Anselin and Hudak (1992), and others have worked on the implementation of spatial estimation and have provided actual code in major languages for computing spatial estimates.

Despite these advances much work needs to be done to extend the benefits of spatial estimation to larger data sets made increasingly available by the widespread use of geographic information systems. Spatial estimators by necessity must examine the relation between each observation and every other observation. This leads to the use of n by n matrices where n represents the number of observations. Quickly, the logistics spiral out-of-control: a 10,000 observation problem creates a 10,000 by 10,000 matrix which would require 800MB of storage (double precision). Computational counts for operations such as determinants and inverses grow with the cube of n. Point estimates alone create these computational exigencies — inference can further exacerbate the computational demands.

As an additional problem, traditional maximum likelihood techniques require non-linear optimization techniques using either analytic derivatives or finite difference approximations. Unfortunately, these can fail to find the global optimum and do so without informing the user of their failure. For example, Ripley (1988, p. 11-15) provides an example with numerous local optima in a one-parameter profile likelihood problem.

Hence, an ideal spatial estimator would (a) handle large data sets; (b) handle point estimation and inference quickly; and (c) not rely on local non-linear optimization algorithms. This paper provides algorithms which achieve all of these goals.

The main weapon against these problems is sparseness, the prevalence of zeros in a matrix. The zeros arise because only nearby observations directly affect each other. Sparseness greatly accelerates computations and reduces storage requirements. For example, sparseness allows us to compute 100 determinants of 3107 by 3107 matrices in 83 seconds. Moreover, while the dense version of these matrices would require 77MB, the sparse version requires less than 1MB. A secondary method is based on reorganizing the computations in the likelihood to avoid iteration and to solve equations as opposed to computing inverses.

We examine the case where the dependent variable follows a spatial autoregressive process.[1] This case can subsume the autoregressive process in errors case when the independent variables include variables with their spatial lags (Anselin 1988, p. 227).[2]

To illustrate our technique, we provide an example and a simulation using U.S. counties or their equivalents. Counties have many attractive characteristics as a geographical entity. First, they tend to have stable definitions over recent time in contrast to census tracts and blocks. Second, many variables are available at this level of aggregation. For example, *USA Counties on CD-ROM* contains 2,844 variables across 3,141 counties or their equivalents across the U.S. These data cover many topics of interest such as age, agriculture, crime, housing, income, education, and elections. More aggregation would destroy the geographical nature of the data and less aggregation often reduces data availability because of problems with ensuring the privacy of respondents.

As our empirical example, we model voting behavior across space. Many countries have automatic voter registration and compulsory voter turnout. In contrast, the relatively low rate of voter registration and turnout in the U.S. generates considerable comment. Most of the discussion centers around variables such as income, traditionally discussed without explicitly considering spatial effects. However, political offices and parties have traditionally been organized in hierarchies using such geographical entities as precincts, wards, counties, and states. It seems plausible that the effects of voting across space cannot be completely described by a small number of observable independent variables. To what degree do the observable independent variables versus the spatial effects describe voting behavior across US counties? We show how our techniques facilitate investigating the question of the relative importance of these two components.

In addition, we conduct a Monte Carlo experiment involving 22,500 regressions of 3107 observations per regression using the same spatial weight matrix employed in the empirical voting example. The simulation demonstrates how sparsity greatly reduces the difficulty of generating vectors of spatially dependent variates. As we show, it takes, on average, only 0.1 second each using the Matlab language on a 200Mhz Pentium Pro to generate a spatially dependent vector and to find the resultant estimates.

The next section presents the model, likelihood function, and estimation procedures. Latter sections show the role of sparsity, illustrate the techniques with a voting example, and discuss a Monte Carlo study of the estimator. The final section presents the key results.

A QUICK SPATIAL AUTOREGRESSIVE ESTIMATOR

This overall section discusses estimation of a spatial autoregressive process in the dependent variable. Part 1 presents the model and its likelihood, part 2 discusses the estimated generalized least squares (EGLS) and maximum likelihood (ML) estimation of the model, while part 3 provides a means to conduct inference without computing information matrices, and part 4 examines the computation of likelihood ratio tests.

1. The Spatial Autoregressive Likelihood Function

When the dependent variable exhibits spatial autocorrelation, the simultaneous autoregression estimator corrects the usual prediction of the dependent variable, $Y = X\beta + \varepsilon$, by a weighted average of the values on nearby observations, *DY*.

$$Y - \alpha DY = X\beta + \varepsilon \tag{1}$$

where D represents an n by n weighting matrix with 0s on the diagonal (the observation cannot predict itself) and α represents the autoregressive parameter.[3] We could also rewrite (1) as (2).

$$(1-\alpha)Y + \alpha(I-D)Y = X\beta + \varepsilon \tag{2}$$

Hence, we are looking for some optimal convex combination of Y in levels and its spatial first differences $(I\text{-}D)Y$. To maintain the interpretation of a weighted average, the rows of D sum to 1 as implied by (3) below. Such weighting matrices are said to be row-standardized (Anselin and Hudak 1992, p. 514). A non-zero entry in the jth column of the ith row indicates that the jth observation will be used to adjust the prediction of the ith observation ($i \neq j$). After correcting for these interactions, the simultaneous autoregressive (SAR) models assume the residuals, ε, are independently and normally distributed.[4] These assumptions are summarized as:

$$\begin{aligned}
&\text{(a)}\ \ D_{ii} = 0 \quad \forall i \\
&\text{(b)}\ \ \sum_{j=1}^{n} D_{ij} = 1 \ \ \forall i \\
&\text{(c)}\ \ 0 \leq \alpha < 1 \\
&\text{(d)}\ \ \varepsilon \sim N(0,\ \sigma^2 I)
\end{aligned} \tag{3}$$

As an illustration of how to construct D, compare the distance d_{ij} between every pair of observations j and i to $d_{i:m}$, the distance from observation i and its mth nearest neighbor. It seems reasonable to set to 0 the direct influence of distant observations upon a particular observation. Accordingly, assign a weight of 1 only to observations whenever d_{ij} is greater than 0 and is less than or equal to $d_{i:m}$ as in (4),

$$0 < d_{ij} \leq d_{i:m} \text{ if and only if } w_{ij} = 1. \tag{4}$$

Subsequently, one could normalize the initial weights so that $\sum_{\substack{j=1 \\ i\neq j}}^{n} D_{ij}=1$ thus making it a row-standardized weight matrix.

$$D_{ij}=\frac{w_{ij}}{\sum_{\substack{j=1 \\ i\neq j}}^{n} w_{ij}} \tag{5}$$

Finally, the profile likelihood function for the autoregressive model in (1) appears in (6),

$$L(\beta,\alpha,\sigma^2)=C+\ln|I-\alpha D|-\left(\frac{n}{2}\right)\ln(SSE) \tag{6}$$

where C represents a constant and SSE denotes sum-of-squared errors.[5]

2. Estimated Generalized Least Squares Computations

If one knew the value of α, the generalized least squares estimator (GLS) would unbiasedly estimate β. When the generalized least squares estimator depends upon an estimated parameter, it becomes the estimated generalized least squares estimator (EGLS) which behaves differently than GLS. Unfortunately, EGLS leads to substantial bias in estimating α in a spatial context (Ripley 1981, p. 91). Taking (1) and forming the SSE yields (7).

$$SSE=(Y-\alpha DY-X\beta_{\alpha})'(Y-\alpha DY-X\beta_{\alpha}) \tag{7}$$

Conditional upon α, the optimal solution for β_{α} (Anselin 1988, p. 181) is,

$$\beta_{\alpha}=(X'X)^{-1}X'(I-\alpha D)Y=\beta_{o}-\alpha\beta_{d} \tag{8}$$

where $\beta_{o}=(X'X)^{-1}X'Y$ and $\beta_{d}=(X'X)^{-1}X'DY$. Substituting (8) into the SSE function in (7) yields (9),

$$SSE = (Y - X\beta_o - \alpha DY + \alpha X\beta_d)'(Y - X\beta_o - \alpha DY + \alpha X\beta_d)$$
$$SSE = (e_o - \alpha e_d)'(e_o - \alpha e_d) = e_o'e_o - 2\alpha e_d'e_o + \alpha^2 e_d'e_d \tag{9}$$

where e_o represents the residuals from an ordinary least-squares (OLS) regression of *Y* on *X* and e_d represents the residuals from an OLS regression of *DY* on *X*.

Interestingly, if one simply wishes to compute EGLS one could form the first order conditions as in (10).

$$\frac{\partial SSE}{\partial \alpha} = -2e_d'e_o + 2\alpha e_d'e_d \tag{10}$$

This leads to the simple solution in (11).

$$\alpha_{egls1} = \left(e_d'e_d\right)^{-1} e_d'e_o \tag{11}$$

3. Maximum Likelihood Computations

Returning to the maximum likelihood approach, one could rewrite the log-likelihood function by substituting the expression for the SSE in (9) into the log-likelihood function in (6). After dropping the constant, this yields (12).

$$L(\beta,\alpha,\sigma^2) = \ln|I - \alpha D| - \left(\frac{n}{2}\right)\ln(e_o'e_o - 2\alpha e_d'e_o + \alpha^2 e_d'e_d) \tag{12}$$

We wish to maximize the log-likelihood over α by selecting a vector of length *m* of values over [0,1) which we label $\alpha_v = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and evaluate the log-likelihood at each of the values contained in α_v as shown in (13).

$$\begin{bmatrix} L(\beta,\alpha_1,\sigma^2) \\ L(\beta,\alpha_2,\sigma^2) \\ . \\ . \\ L(\beta,\alpha_m,\sigma^2) \end{bmatrix} = \begin{bmatrix} \ln|I - \alpha_1 D| \\ \ln|I - \alpha_2 D| \\ . \\ . \\ \ln|I - \alpha_m D| \end{bmatrix} - \left(\frac{n}{2}\right) \begin{bmatrix} \ln(e_o'e_o - 2\alpha_1 e_d'e_o + \alpha_1^2 e_d'e_d) \\ \ln(e_o'e_o - 2\alpha_2 e_d'e_o + \alpha_2^2 e_d'e_d) \\ . \\ . \\ \ln(e_o'e_o - 2\alpha_m e_d'e_o + \alpha_m^2 e_d'e_d) \end{bmatrix} \tag{13}$$

Given (a) the scalars $e_o'e_o$, $e'e_o$, and $e'e$ and (b) the vector of log-determinant values associated with α_v, evaluating the likelihood in (13) becomes quite easy. Picking the element of α_v , which yields the maximum value of the vector of log-likelihood values, α_{ml}, requires very little time. Usually, the main difficulty lies in computing $\ln|I-\alpha D|$.

From a computational standpoint, the vectorization of the problem avoids the relatively high overhead incurred by invocation of non-linear optimizers and their normally sequential nature. In computational environments with fast vector operations, such sequential operations can greatly reduce performance.

The use of a finite set of α will cause some small granularity in the chosen values α_{ml} , but it should not prove difficult to make the granularity small relative to the statistical precision of the estimated α_{ml}. While this approach may suffer a small loss of precision relative to non-linear maximization, evaluating the log-likelihood function over a grid offers the advantage of robustness.

4. Avoiding Computation of the Information Matrix

Computation of the information matrix often becomes expensive in a spatial context. Inspection of the information matrix in Ord (1975, p. 124-125) or in Anselin (1988, p. 76-77) shows it requires, among other computations, an *n* by *n* inverse and multiplication of two *n* by *n* matrices. Moreover, its usual formulation requires the eigenvalues of the spatial weight matrix. Finding the eigenvalues of a large matrix demands substantial computational resources.

In addition, the information matrix approach works best when the profile likelihoods are quadratic in α. However, most plots of the profile likelihood display substantial asymmetry (Ripley 1988, p. 14). In such cases, as Meeker and Escobar (1995) and others forcefully argue, profile likelihood techniques can outperform the information matrix approach. Finally, the information matrix approach requires enough “smoothness” to make second derivatives well-behaved.

Fortunately, since the technology previously outlined facilitates rapid maximization of the likelihood, this provides an alternative route to inference. For example, the speedy maximization of the likelihood function allows one to compute restricted least squares estimates for β_o, β_d, maximize the likelihood, and form a likelihood ratio test for each hypothesis.

In (14) appears the formula for the restricted least-squares estimator given the linear hypotheses $R\beta_\alpha = r$ (Judge *et al.* 1988, p. 235-264).

$$\beta_{\alpha-rest} = \beta_\alpha + (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - R\beta_\alpha) \qquad (14)$$

For a single hypothesis such as $\beta_2 = 0$, R is 1 by k, r is a scalar, the expression $[R(X'X)^{-1}R']^{-1}$ reduces to a scalar, and evaluation of $\beta_{\alpha-rest}$ occurs rapidly. For example, one could evaluate $\beta_j = 0$ for each *j* separately. This would yield a likelihood ratio test for the significance or lack of significance of each variable which would supplant the need to compute the information matrix for forming the usual asymptotic *t* statistics.

More generally, for a joint hypothesis comprised of *h* parts, R is h by k, r is a h by 1 vector, the expression $[R(X'X)^{-1}R']^{-1}$ reduces to an *h* by *h* matrix. Provided the number of parts of the joint hypothesis, *h*, is substantially smaller than the number of variables, *k*, evaluation of $\beta_{\alpha-rest}$ proceeds quite rapidly.

As a more detailed example, suppose interest centers on whether the variable X_2 and its spatial lag DX_2 significantly affect the overall regression involving 3 basic variables and their spatial lags for a total of 6 variables (assume mean centering to avoid an intercept). This question leads to the joint hypothesis $\beta_2 = 0$ and $\beta_4 = 0$. Hence, the matrix R and the vector r appear as,

$$R = \begin{array}{c} \begin{matrix} X_1 & X_2 & X_3 & DX_1 & DX_2 & DX_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

While one could drop variables to implement hypotheses such as $\beta_j = 0$, the use of restricted least squares avoids recomputing the moment matrix ($X'X$) and its inverse ($(X'X)^{-1}$), a relatively lengthy task when *n* and *k* become large.[6] In addition, restricted least-

squares can handle more general specifications such as the joint hypothesis
$\beta_2 = 1$ and $\beta_4 = 1 - \beta_3$.

SPARSITY AND COMPUTATIONS

If differencing an observation with its nearby neighbors removes most of the effects of autocorrelation, the spatial weighting matrix *D* can be quite sparse. For example, if an observation displays error dependency with its nearest *m* neighbors, only *m* non-zero entries exist per row of *D*. Thus, *D* will contain *nm* non-zero elements out of n^2 possible elements. This produces a *m/n* proportion of non-zero elements, a popular measure of sparsity. For example, with this problem we used four neighbors for each observation. Hence, *D* has sparsity of 4/3107 (0.13%). This represents a very high level of sparsity which increases as *n* grows.

Sparsity results in a number of computational gains. First, it dramatically decreases the storage needed for *D* and $(I - \alpha D)$. Using traditional dense techniques, *D* requires 77.28MB of storage (double precision). Using sparse matrix techniques, *D* requires less than 1MB of storage. Naturally, this divergence grows with *n*. The use of sparse matrix technology has allowed us to handle problems with 20,640 observations (Pace and Barry forthcoming).

Second, sparsity greatly accelerates computations. For example, multiplying the *n* by *n* matrix *D* by the *n* by *k* matrix *X* requires $O(kn^2)$ operations using dense matrices. Barring computational bookkeeping, the equivalent sparse operation requires $O(knm)$ operations. The real benefits come when computing determinants, inverses, or solving systems of equations. All of these operations can build upon the LU decomposition of a matrix through Gaussian elimination and all require $O(n^3)$ operations when using dense matrices (Golub and Van Loan 1989). Sparsity, however, can totally change the order of the number of operations required in these computations. For example, if $(I - \alpha D)$ had a band structure with lower bandwidth *p* and upper bandwidth *q*, the LU decomposition of

$(I-\alpha D)$, would require O(2*npq*) operations (Golub and Van Loan 1989, p. 151). Hence, for fixed bandwidths the computations grow linearly with *n*, the number of observations.

Unfortunately, the existence of a pure band structure does not arise very often. Figure 1a shows the actual plot of the non-zero elements in $(I-\alpha D)$. Each of the blocks on the diagonals represent states. The dispersed off-diagonal elements represent counties contiguous with those from a different state. The existence of such dispersed off-diagonals could make it difficult to achieve computational gains. However, one can permute the rows or columns of $(I-\alpha D)$ using the reverse Cuthill-McKee algorithm to create a variable band matrix as shown in Figure 1b. Figure 1b makes the gains of exploiting sparsity obvious. Less obviously, Figure 1c shows the plot of $(I-\alpha D)$ permuted using the column minimum degree algorithm. Counterintuitively, this ordering usually produced the fastest LU decomposition times. See George and Liu (1981) for a discussion of the reverse Cuthill-McKee, minimum degree, and other orderings useful in accelerating the computation of matrix decompositions.

While the actual mechanics of these algorithms may seem quite involved, the intuition is simple. If one had many equations to solve, the fewer variables in each equation the better (more sparsity preferred). It seems intuitive to arrange the equations so that each one uses variables present only in nearby equations (low bandwidth preferred). Ideally, one would like to order the equations so that one could solve the first one for a variable, then take this variable's value, substitute it into the second one, solve for the second variable, and so on. Gaussian elimination and the LU decomposition allow one to perform precisely this procedure. Hence, these algorithms formalize and swiftly execute a natural set of computations.

We computed $\ln|I-\alpha D|$ for 100 values of α over [.005, .015, ... .995]. Figure 2 shows the plot of the log-determinants versus α. Table 1 shows the timings associated with computing the LU decomposition using the original, random, reverse Cuthill-McKee, and column minimum degree orderings for $(I-\alpha D)$. As Table 1 makes clear, the ordering of the rows and columns matter, with the column minimum degree ordering reducing execution times by 91.98% over the original ordering. The column minimum degree ordering also reduced execution times by 58.71% over the more intuitive reverse Cuthill-McKee ordering. As a worst case scenario, the random ordering produced computational times worse than the optimal ordering by a factor of 105.95. All computations used the Matlab language running on a 200Mhz Pentium Pro computer.

To place these results in perspective, Li (1995) took the eigenvalue route to computing determinants. Li used an IBM RS6000 Model 550 and a CM5 parallel processing supercomputer. The CM5 had 32 processors each with 32MB of local memory and four vector units. For a 2500 by 2500 spatial weight matrix the RS6000 required 8515.07 seconds while the CM5 required 45.78 seconds. Adjusting for size differences ($(3107/2500)^3$) these times would go to 16345.26 and 87.88 seconds for a 3107 by 3107 problem. Hence, the use of sparse technology allows personal computers to approach supercomputer performance for this problem.

The use of sparsity does not preclude the use of supercomputing technology. A substantial amount of development has gone into devising parallel sparse routines (Saad 1996, p. 324-422). Employing supercomputers and sparseness could vastly extend the range of computable spatial problems. Moreover, as demonstrated herein, one can easily vectorize spatial estimators.

Note, the smoothness of $|I - \alpha D|$ suggests we could evaluate it over fewer values of α or concentrate the computations around α likely to occur. We have achieved good performance with the use of a 20th degree polynomial, which would reduce the number of determinant computations above by a factor of five.

VOTING ACROSS COUNTIES

In this section we illustrate the techniques presented in previous sections using data on the votes cast in the 1980 presidential election across U.S. counties. In what follows, part 1 discusses the data employed, part 2 presents the two models used, part 3 gives the estimation results, while part 4 demonstrates the use of the likelihood ratio tests discussed in a previous section.

1. Voting Data

Specifically, we used the geographic centroids from all the counties (or their equivalents) in the continental U.S. from the 1990 Census which recorded votes in the 1980 presidential election. This yielded a matrix D with 3,107 rows and 3,107 columns.[7] We picked 1980 because the presidential election cycle of every four years corresponded to the census data collection cycle of every ten years.

We collected data on the total number of votes cast in the 1980 presidential election per county (VOTES), the population in each county of 18 years of age or older (POP), the population in each county with a 12th grade or higher education (EDUCATION), the number of owner-occupied housing units (HOUSES), and the aggregate income (INCOME).

2. Models

We fitted two models by OLS and maximum likelihood, respectively. We elected to examine the log of the proportion of votes cast for both candidates in the 1980 presidential election. Hence, we can express our dependent variable as ln(PRVOTES)= ln(VOTES/ POP) = ln(VOTES)-ln(POP). We fitted the following model via OLS:

$$\begin{aligned}\ln(\text{PRVOTES}) = {} & \text{Intercept}\beta_1 + \ln(\text{POP})\beta_2 + \ln(\text{EDUCATION})\beta_3 + \ln(\text{HOUSES})\beta_4 \\ & + \ln(\text{INCOME})\beta_5 + \varepsilon\end{aligned}$$

We fitted the following model which subsumes the previous model via maximum likelihood:

$$\begin{aligned}\ln(\text{PRVOTES}) = {} & \text{Intercept}\beta_1 + \ln(\text{POP})\beta_2 + \ln(\text{EDUCATION})\beta_3 + \ln(\text{HOUSES})\beta_4 \\ & + \ln(\text{INCOME})\beta_5 + D\ln(\text{POP})\beta_6 + D\ln(\text{EDUCATION})\beta_7 + D\ln(\text{HOUSES})\beta_8 \\ & + D\ln(\text{INCOME})\beta_9 + D\ln(\text{PRVOTES})\,\alpha + \varepsilon\end{aligned}$$

This looks at the same fundamental model, but adds to it the spatial lags of the dependent and independent variables.

3. Estimation Results

As Table 2 documents, both the OLS and maximum likelihood predictions showed reasonably good relative fit with R^2s of .5242 and .7123. As this indicates, maximum likelihood using the spatial information displayed considerably lower error than OLS. In fact, the SSE from OLS of 49.2825 was 78.12% higher than the SSE of 27.6686 from maximum likelihood. Moreover, the median absolute error from OLS of .0864 was 40.49% higher than the median absolute error of .0615 from maximum likelihood.

The OLS estimates displayed the expected signs with the exception of income which was negative and significant. The variable ln(POP) had a negative sign for both OLS and maximum likelihood. However, this arises because we use the proportion of votes cast which effectively removes a coefficient of 1 from both sides. Given the measured coefficients on ln(POP) were greater than -1, population has the expected positive overall effect on votes but the proportion voting declines with population *per se*. Similarly, maximum likelihood on the expanded model displayed the expected signs on all the variables. Inspection of the maximum likelihood results show the probable source of OLS's difficulties. The significance of spatially lagged income, which the OLS model omits, and its correlation with income, led OLS to adjust towards the negative and significant omitted variable.

Relative to OLS, maximum likelihood ascribes a smaller influence of education and a larger influence of home ownership upon the propensity to vote. Naturally, the large coefficient on the lagged dependent variable suggests the existence of geographically correlated but omitted variables.

4. Inference

The second section discussed means of avoiding computation of the information matrix in a spatial setting. When the likelihood costs little to compute, likelihood ratio tests have substantial appeal. In this case, the restricted maximum likelihood estimates do not require much computational resources. We test the hypotheses that each of the basic independent variables and their spatial lags have no effect upon the regression. As we have nine total independent variables (four basic variables, four spatially lagged basic independent variables, and an intercept), the matrix R has 4 rows and 9 columns. It contains all zeros except for a one in each row in the position of the variables whose effects we wish to set to zero. The vector r is a 4 by 1 vector of zeros.

Table 3 displays the results of the likelihood ratio tests for the deletion of all spatially lagged variables, of the spatially lagged dependent variable, and all of the basic independent variables with their lags. All of the variables or combinations of the variables were statistically significant. It required only .09 seconds to compute the restricted likelihoods for the deletion of the basic independent variables with their lags using the techniques presented earlier.

The likelihood associated with the OLS regression on the basic variables coupled with the maximum likelihood obtained for all variables give the likelihood ratio test for the significance of all spatially lagged variables. The likelihood associated with $\alpha=0$ from the profile likelihood coupled with the maximum likelihood for all variables obtained give the likelihood ratio test for the significance of the spatially lagged dependent variable, Dy. These likelihoods require no additional computations but appear as a byproduct from the basic procedure and hence impose little computational cost.

The significance of α, the parameter estimate of the spatially lagged dependent variable, and the significance of the other spatially lagged independent variables shows the substantial contribution geographically correlated variables make to the overall fit.

AN ILLUSTRATIVE SIMULATION

This section examines a simulation of 22,500 regressions each using 3107 observations. Part 1 discusses the data, part 2 gives the timings of the simulation computations, and part 3 provides the statistical results.

1. Simulation Data

To provide verisimilitude to the simulation, we chose the same spatial weighting matrix as employed in the empirical voting example.

In the simulation, we:

1. Generated uniform random variables for nine columns of X and used a constant for the other column.
2. Set β to a vector of ones.
3. Let α_v equal [.01, .05, .1, .25, .5, .75, .9, .95, .99].
4. Set σ to each of [.1, .5, 1, 2, 10].
5. Generated a common set of 250 N(0,1) vectors of 3107 elements each using the Matlab normal random number generator. We perform this operation once for the entire simulation. Scaling the common N(0,1) errors by σ generates the N(0, σ^2) random variables. This practice, referred to as "correlated sampling" (Rubinstein (1981)) greatly reduces the variance in Monte Carlo experiments.

We subsequently generated the autocorrelated dependent variable, y, according to (15),

$$y = (I - \alpha D)^{-1} X\beta + \sigma(I - \alpha D)^{-1} u \qquad (15)$$

where u represents an n by *iter* matrix of N(0,1) random variates where *iter* represents the number of iterations in the experiment. In actuality, we solved the corresponding equation system in (16) for the coefficients Z via Gaussian elimination using an LU decomposition rather than computing the inverse as this goes much faster.

$$\begin{aligned}&\underset{n\times n}{(I-\alpha D)}\underset{n\times(iter+1)}{Z}=\underset{n\times(iter+1)}{\left[X\beta \quad u\right]}\\&Z=\left[(I-\alpha D)^{-1}X\beta \quad (I-\alpha D)^{-1}u\right]\end{aligned}\tag{16}$$

Compare this to the usual inverse formulation,

$$\begin{aligned}&\underset{n\times n}{(I-\alpha D)}\underset{n\times n}{Q}=\underset{n\times n}{I}\\&Q=\underset{n\times n}{(I-\alpha D)^{-1}}\\&\underset{n\times(iter+1)}{Z}=\underset{n\times n}{Q}\underset{n\times(iter+1)}{\left[X\beta \quad u\right]}=\underset{n\times n}{(I-\alpha D)^{-1}}\underset{n\times(iter+1)}{\left[X\beta \quad u\right]}=\left[(I-\alpha D)^{-1}X\beta \quad (I-\alpha D)^{-1}u\right]\end{aligned}\tag{17}$$

Relative to (16), (17) requires solving a larger system (n by n instead of n by $(iter+1)$) and subsequently multiplying an n by n matrix by a n by $iter+1$ matrix ($O(n^2(iter+1))$). Since $iter+1$ usually is much smaller than n, the inverse method takes substantially longer to yield the same results.

2. Timing Results from the Monte Carlo Experiment

Table 4 contains the results from the Monte Carlo experiment using the autoregressive dependent variable model. Table 4 contains 45 cases resulting from nine autoregressive parameter values, α, and five levels of error variability, σ. Each case contains the average of the results from five runs of 100 iterations each. The runs required 36.8 minutes in total. Thus, each maximum likelihood spatial autoregression and simulated dependent variable needed 0.1 seconds of computational time, a very low figure.

3. Statistical Results from the Monte Carlo Experiment

The results in Table 4 match some of those reported in the literature using regular lattices.[8] First, the maximum likelihood estimator slightly underestimates the true differencing parameter, α. Second, the EGLS estimator overestimates α. Note, two versions of the EGLS estimator appear. The first one comes from (11). The second one comes from picking the value of α off the grid which minimizes SSE. Hence, one could consider EGLS$_2$ an inequality restricted EGLS estimator. It has the same granularity problem as the maximum likelihood estimator and hence makes a better comparison with the maximum likelihood estimator. For example, examine the cases where α equals 0.99. In these cases, both the $EGLS_2$ and ML give the same value of the square root of the mean squared error (RMSE) of 0.05. However, as the comparison between $EGLS_1$ and $EGLS_2$ reveals, for most cases the granularity problem does not greatly affect the results. The inequality restricted nature of $EGLS_2$ does lead to it performing differently than $EGLS_1$ at the endpoints.

Third, the maximum likelihood estimator greatly outperforms EGLS in most cases. In fact, maximum likelihood displays a factor of 14.81 better performance at the worst case for EGLS (α=.5, σ=10). EGLS becomes most acceptable for high α and low σ. Fourth, all of the estimators perform worse as σ rises.

Each estimator reaches the nadir of its performance at various points depending upon both α and σ. For example, the maximum likelihood estimator reaches its nadir at α of 0.1 with σ of 10. The $EGLS_1$ estimator reaches its nadir at α of 0.5 with σ of 10.

CONCLUSION

A variety of methods can greatly accelerate the computation of large scale spatial autoregressions. This paper explored the use of sparse matrix techniques, formulating the profile likelihood to avoid iterative computations, and using restricted least squares to form likelihood ratio tests as opposed to computing information matrices. As illustrations of the efficacy of these techniques, we looked at an empirical example at the county level and conducted a Monte Carlo experiment with 22,500 spatial autoregressions. Despite the formidable size of the regressions (3107 observations), these cost only 0.1 seconds each to generate the spatially dependent variable and estimate the coefficients.

The voting example showed the power of geographic information to help clarify social phenomenon. OLS on the non-spatial variables displayed 78% higher sum-of-squared errors than maximum likelihood on the combination of spatial and non-spatial variables. Relative to OLS, maximum likelihood ascribes a smaller influence of education and a larger influence of home ownership upon the propensity to vote. More importantly, OLS shows income has a significant, negative effect on the proportion voting while maximum likelihood shows income has a small positive but statistically insignificant effect. However, spatially lagged income has a significant, negative effect upon voting.

Hopefully, additions like these to the spatial statistics toolkit will allow computations to keep pace with the ever-increasing flow of geographic information and bring spatial techniques into more routine usage.

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Table 1 — Execution Times in Seconds Versus Ordering Algorithm

Ordering Algorithm	LU $(I - \alpha D)$
Original	10.35
Reverse Cuthill-McKee	2.01
Column Minimum Degree	0.83
Random	87.94

Table 2 — OLS and ML Estimates of Voting Characteristics

	B_{ols}	t_{ols}	B_{ml}	t_{ml}
Intercept	0.9814	20.9680	0.4582	10.2893
ln(Population > 18 years of Age)	-0.8464	-38.3780	-0.7174	-31.3059
ln(Population with Education > 12 years)	0.5167	33.4241	0.1910	8.1971
ln(Owner Occupied Housing Units)	0.4291	24.0222	0.4513	28.8363
ln(Aggregate Income)	-0.1439	-7.2680	0.0266	1.2089
Lagged ln(Population > 18 years of Age)			0.3999	-13.2999
Lagged ln(Population with Education > 12 years)			0.0513	1.9398
Lagged ln(Owner Occupied Housing Units)			-0.2906	-12.9525
Lagged ln(Aggregate Income)			-0.1252	-4.5015
Optimal α			0.6150	
R^2	0.5242		0.7123	
SSE	49.2825		27.6686	
Median $\lvert e \rvert$	0.0864		0.0615	
Log(Likelihood)	-6307.1		-5679.1	
n	3107		3107	
k	5		10	
Time	.03 sec		.09 sec	

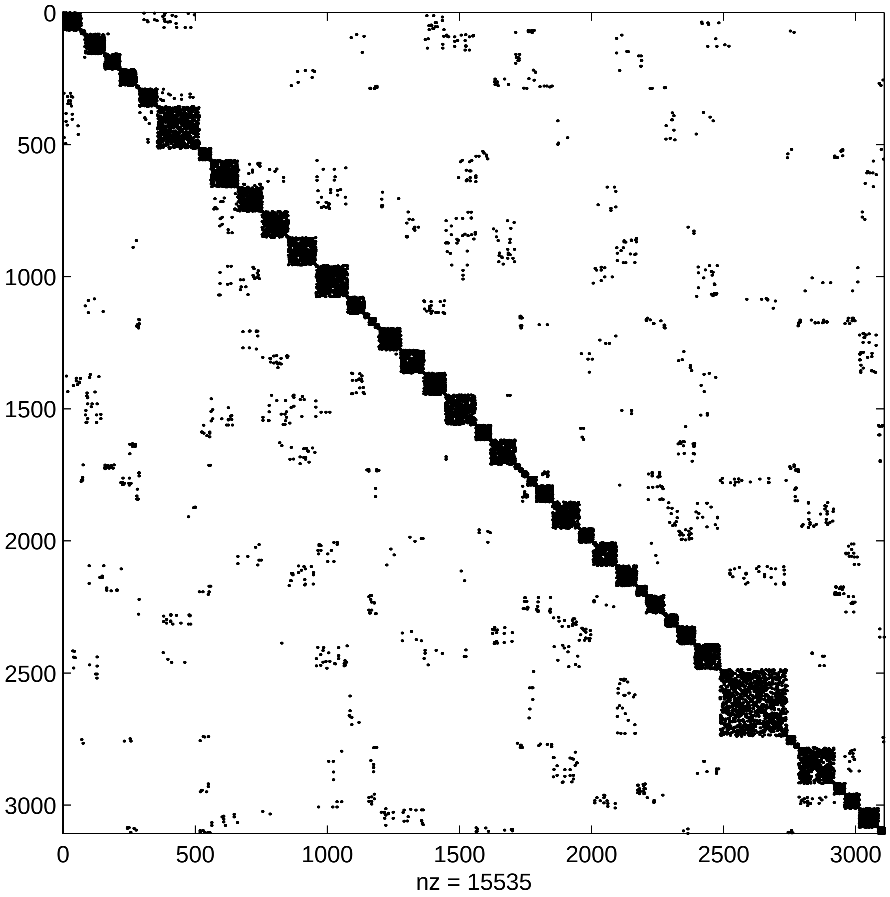
Table 3 — Likelihood Ratio Tests for the Deletion of Different Variables

Deletion of:	Unrestricted Likelihood	Restricted Likelihood	Likelihood Ratio	Number of Hypotheses
All Spatially Lagged Variables	-5679.0911	-6307.0766	1255.9710	5
Dy	-5679.0911	-6222.4552	1086.7282	1
ln(Population > 18 years of Age)	-5679.0911	-6120.5844	882.9866	2
ln(Population with Education > 12 years)	-5679.0911	-5809.8701	261.5580	2
ln(Owner Occupied Housing Units)	-5679.0911	-6045.2873	732.3924	2
ln(Aggregate Income)	-5679.0911	-5692.6655	27.1488	2

Table 4 — True Vs. Estimated Parameters and RMSEs Across 500 Autoregressions

α	σ	α_{ml}	α_{egls2}	α_{egls1}	$RMSE_{ml}$	$RMSE_{egls2}$	$RMSE_{egls1}$
0.01	0.1	0.0104	0.0105	0.0102	0.0051	0.0052	0.0042
0.01	0.5	0.0140	0.0172	0.0121	0.0116	0.0154	0.0201
0.01	1.0	0.0165	0.0236	0.0140	0.0155	0.0253	0.0336
0.01	2.0	0.0174	0.0284	0.0153	0.0172	0.0330	0.0438
0.01	10.0	0.0175	0.0307	0.0158	0.0179	0.0375	0.0493
0.05	0.1	0.0503	0.0510	0.0507	0.0053	0.0053	0.0042
0.05	0.5	0.0497	0.0605	0.0605	0.0163	0.0226	0.0223
0.05	1.0	0.0496	0.0733	0.0731	0.0220	0.0398	0.0401
0.05	2.0	0.0493	0.0841	0.0835	0.0244	0.0537	0.0542
0.05	10.0	0.0493	0.0896	0.0889	0.0252	0.0610	0.0617
0.10	0.1	0.1002	0.1013	0.1012	0.0053	0.0053	0.0043
0.10	0.5	0.0996	0.1209	0.1207	0.0160	0.0285	0.0282
0.10	1.0	0.0993	0.1462	0.1464	0.0222	0.0561	0.0563
0.10	2.0	0.0987	0.1674	0.1677	0.0249	0.0790	0.0792
0.10	10.0	0.0986	0.1791	0.1791	0.0259	0.0917	0.0917
0.25	0.1	0.2503	0.2528	0.2527	0.0052	0.0056	0.0047
0.25	0.5	0.2494	0.2982	0.2983	0.0154	0.0513	0.0514
0.25	1.0	0.2491	0.3594	0.3591	0.0205	0.1130	0.1127
0.25	2.0	0.2487	0.4093	0.4094	0.0234	0.1632	0.1634
0.25	10.0	0.2483	0.4367	0.4368	0.0245	0.1910	0.1910
0.50	0.1	0.5001	0.5043	0.5041	0.0050	0.0056	0.0052
0.50	0.5	0.4997	0.5771	0.5771	0.0129	0.0782	0.0782
0.50	1.0	0.4988	0.6731	0.6731	0.0170	0.1742	0.1742
0.50	2.0	0.4984	0.7514	0.7514	0.0192	0.2525	0.2525
0.50	10.0	0.4980	0.7940	0.7937	0.0199	0.2951	0.2948
0.75	0.1	0.7500	0.7546	0.7537	0.0050	0.0050	0.0043
0.75	0.5	0.7494	0.8197	0.8197	0.0088	0.0702	0.0701
0.75	1.0	0.7487	0.9055	0.9054	0.0114	0.1559	0.1558
0.75	2.0	0.7482	0.9738	0.9742	0.0128	0.2241	0.2245
0.75	10.0	0.7483	0.9944	1.0109	0.0135	0.2445	0.2612
0.90	0.1	0.9000	0.9046	0.9021	0.0050	0.0050	0.0023
0.90	0.5	0.8993	0.9388	0.9387	0.0056	0.0391	0.0388
0.90	1.0	0.8992	0.9864	0.9865	0.0067	0.0866	0.0866
0.90	2.0	0.8988	0.9950	1.0248	0.0074	0.0950	0.1250
0.90	10.0	0.8988	0.9950	1.0453	0.0077	0.0950	0.1455
0.95	0.1	0.9498	0.9545	0.9511	0.0050	0.0050	0.0013
0.95	0.5	0.9493	0.9727	0.9717	0.0050	0.0231	0.0218
0.95	1.0	0.9492	0.9949	0.9987	0.0052	0.0449	0.0488
0.95	2.0	0.9491	0.9950	1.0204	0.0054	0.0450	0.0707
0.95	10.0	0.9488	0.9950	1.0320	0.0056	0.0450	0.0823
0.99	0.1	0.9890	0.9935	0.9902	0.0050	0.0050	0.0003
0.99	0.5	0.9888	0.9950	0.9945	0.0050	0.0050	0.0046
0.99	1.0	0.9883	0.9950	0.9998	0.0050	0.0050	0.0100
0.99	2.0	0.9885	0.9950	1.0038	0.0050	0.0050	0.0141
0.99	10.0	0.9883	0.9950	1.0059	0.0050	0.0050	0.0162

Figure 1a − Original Ordering



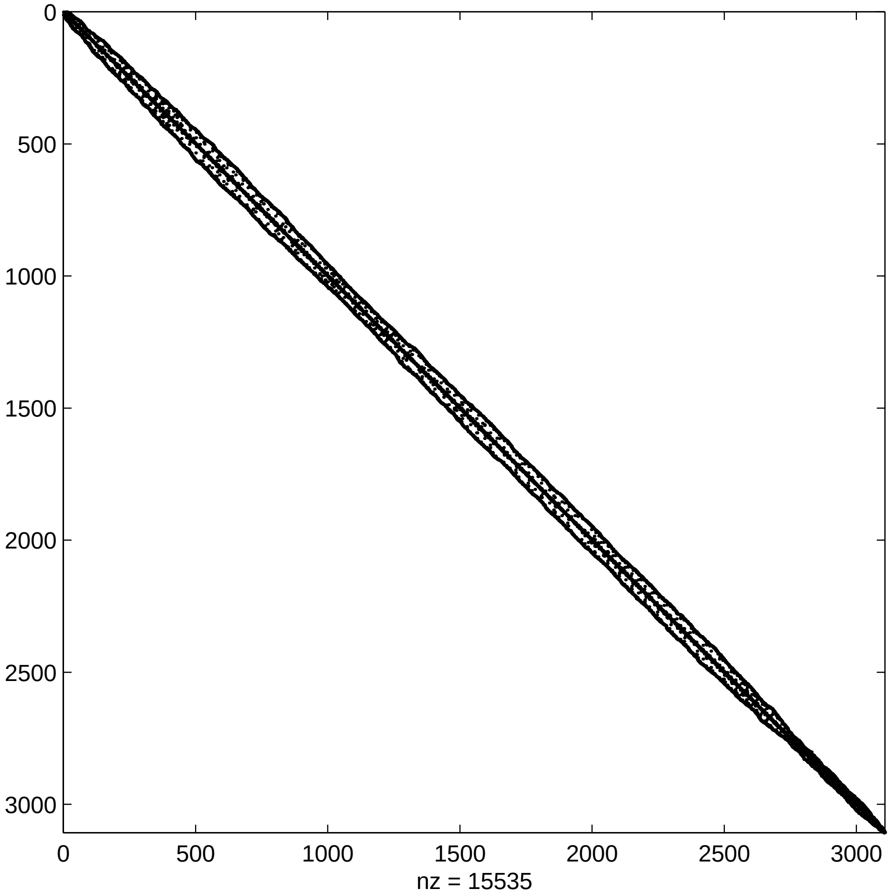


Figure 1b − Reverse Cuthill−McKee Ordering

Figure 1c − Column Minimum Degree Ordering

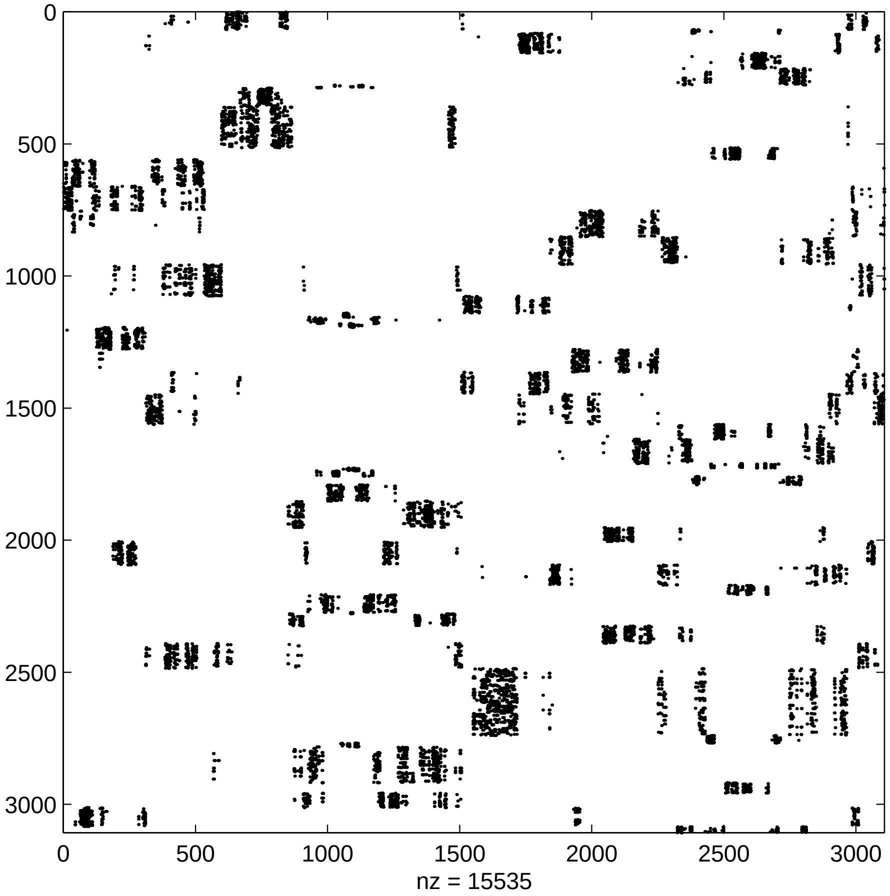
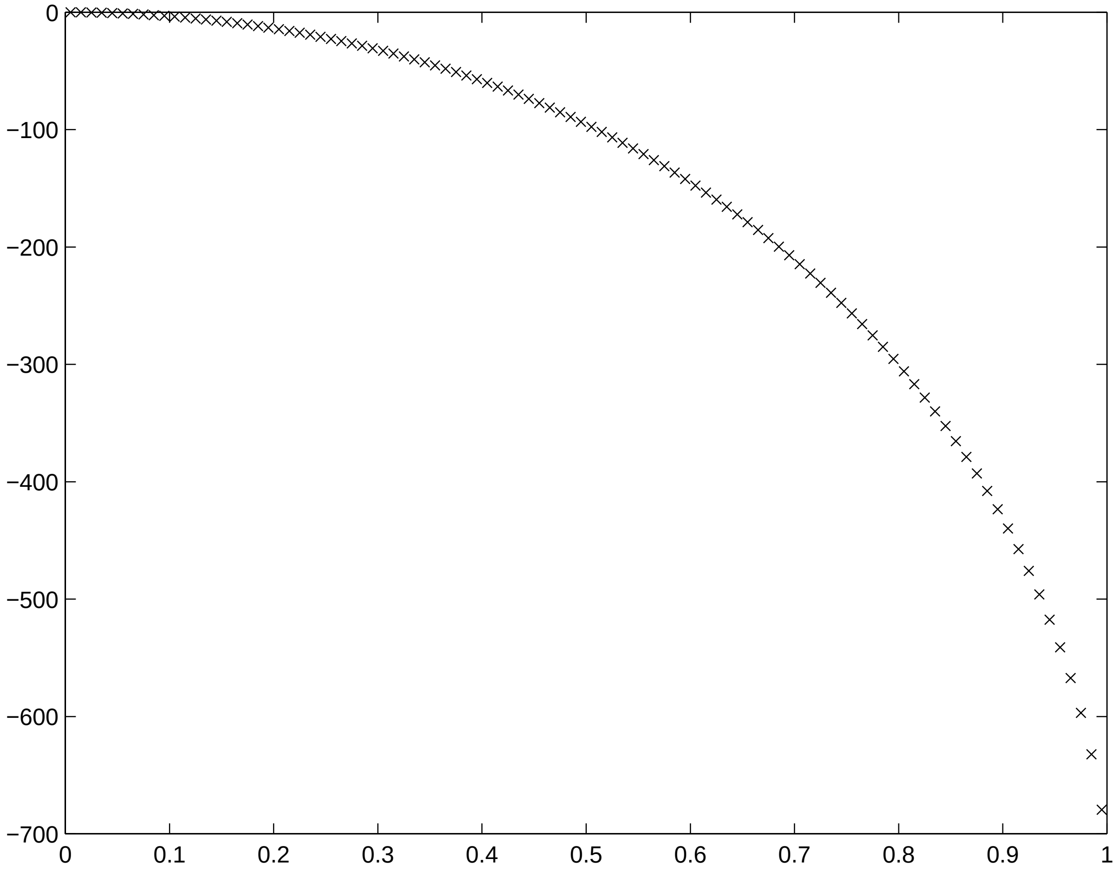


Figure 2



[1] See Can (1992) and Can and Megbolugbe (forthcoming) for examples using the spatial autoregressive dependent variable process.

[2] See Dubin (1988) and Pace and Gilley (forthcoming) for examples using the spatial autoregressive error process.

[3] One could construct X to include $[X_{sub} \mid DX_{sub}]$ in which case this model would subsume the SAR autoregressive error model as a special case.

[4] Specification of spatial autoregressions follow either the simultaneous (SAR) or conditional (CAR) approaches. These lead to similar likelihood functions for normal errors. Geographical applications employ the SAR approach more often than the CAR approach (Anselin 1988, p. 32-33).

[5] See Anselin (1988, p. 182). Our constant includes an extra term as we use SSE instead of the estimated variance.

[6] This suggests that one could extend the spatial autoregressive dependent variable estimator to handle inequality restrictions. See Pace and Gilley (1993) and Gilley and Pace (1995) for more information on inequality restrictions in regression settings.

[7] Three counties in the continental U.S. showed zero votes in the election. These were unusual entities such as Yellowstone National Park.

[8] Haining (1990, p. 135-141) discusses some simulation results on regular lattices. Griffith (1995) conducted some recent simulation studies.